

ARONSON-BÉNILAN ESTIMATES FOR THE POROUS MEDIUM EQUATION UNDER THE RICCI FLOW

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ABSTRACT. In this paper we study the porous medium equation (PME) coupled with the Ricci flow on complete manifolds with bounded nonnegative curvature operator. In particular, we derive Aronson-Bénilan and Li-Yau-Hamilton type differential Harnack estimates for positive solutions to the PME with a linear forcing term.

1. INTRODUCTION

Differential equations of the form

$$\frac{\partial u}{\partial t} = \Delta u^p \quad (1.1)$$

for $p > 0$ are of great interest due to their importance in mathematics, physics, and applications in many other fields. For $p = 1$ it is the well-known heat equation. When $p > 1$, equation (1.1) is known as the porous medium equation (PME), which models the flow of gas through porous medium, ground water filtration, heat radiation in plasmas, etc. We refer the readers to [18] for basic theory and various applications of the porous medium equation on the Euclidean space. In the case where $0 < p < 1$, equation (1.1) becomes the so-called fast diffusion equation (FDE).

In 1979, Aronson and Bénilan [1] obtained an important second order differential inequality

$$\sum_i \frac{\partial}{\partial x_i} (p u^{p-2} \frac{\partial u}{\partial x_i}) \geq -\frac{\kappa}{t}, \quad (1.2)$$

where $\kappa = \frac{n}{2+n(p-1)}$, for any positive solution u to (1.1) on the Euclidean space \mathbb{R}^n with $p > (1 - \frac{2}{n})^+$. On the other hand, for any complete Riemannian manifold (M^n, g_{ij}) with Ricci curvature bounded from below by $-K$ for some $K \geq 0$, Li and Yau [12] discovered the following celebrated differential Harnack estimate, now widely called the Li-Yau estimate,

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}, \quad (1.3)$$

where $\alpha > 1$, for any positive solution u to the heat equation on M^n . Moreover, in the special case where $K = 0$, there holds the sharp Li-Yau estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}. \quad (1.4)$$

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In the same paper, Li and Yau showed that significant results can be obtained by applying the differential Harnack estimates above, such as the classical parabolic Harnack inequality, Gaussian upper and lower bounds of the heat kernel, estimates of the Green's function, and estimates of the Dirichlet and Neumann eigenvalues of the Laplace operator.

Subsequently, matrix differential Harnack estimates for the heat equation were proved by Hamilton [10] on Riemannian manifolds, and by Ni and the first author [3] on Kähler manifolds.

Unlike the study of the heat equation, PME on Riemannian manifolds were only investigated recently. In [18], Aronson-Bénilan and Li-Yau type estimate was first proved by Vázquez for positive solutions to PME on complete manifolds with nonnegative Ricci curvature. More recently, under the weaker assumption that the Ricci curvature of M is bounded from below by $-K$ for some $K \geq 0$, Lu, Ni, Vázquez and Villani [13] derived more general Aronson-Bénilan and Li-Yau type gradient estimates for PME and some FDE. In particular, for any $\alpha > 1$ and any smooth bounded positive solution u to PME, they proved

$$\alpha \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \geq -(p-1)\kappa\alpha^2 \left[\frac{1}{t} + \frac{(p-1)}{\alpha-1} K v_{max} \right], \quad (1.5)$$

where $v = \frac{p}{p-1}u^{p-1}$, $\kappa = \frac{n}{2+n(p-1)}$, and $v_{max} = \max_{M \times [0, T]} v$. See also the related work in [11].

The study of differential Harnack estimates for parabolic equations under the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (1.6)$$

on complete manifolds was originated from Hamilton's work. In [8], for the Ricci flow on Riemann surfaces with positive curvature, he showed a Li-Yau type inequality for the evolving scalar curvature of the Ricci flow.¹ In higher dimensions, both matrix and trace differential Harnack estimates, also known as Li-Yau-Hamilton estimates, were obtained by Hamilton [9] for the Ricci flow with bounded and nonnegative curvature operator. These estimates played a crucial role in the study of singularity formations of the Ricci flow on three manifolds and the Hamilton-Perelman solution to the Poincaré conjecture. The Li-Yau-Hamilton estimate for the Kähler-Ricci flow with nonnegative holomorphic bisectional curvature was obtained by the first author [2]. In addition, in [16] Perelman obtained a Li-Yau type estimate for the fundamental solution of the conjugate heat equation under the Ricci flow (see also [14]).

Recently, by using Hamilton's trace differential Harnack estimate [9], X. Cao and Hamilton [5] proved the Li-Yau-Hamilton type estimate

$$2\Delta v - |\nabla v|^2 - 3R - \frac{2n}{t} \leq 0 \quad (1.7)$$

on compact manifolds, where $v = -\ln u$ and u is any positive solution to the forward conjugate heat equation

$$\frac{\partial u}{\partial t} = \Delta u + Ru \quad (1.8)$$

¹Later, Chow [6] was able to remove the positivity assumption and obtained a similar estimate.

under the Ricci flow with nonnegative curvature operator. Also, certain matrix differential Harnack estimate was shown by Ni [15] for positive solutions of (1.8) under the Kähler-Ricci flow.

In this paper, we consider equation of type (1.1) with a linear forcing term,

$$\frac{\partial u}{\partial t} = \Delta u^p + Ru, \quad (1.9)$$

under the Ricci flow on a complete manifold M^n and prove Aronson-Bénilan and Li-Yau-Hamilton type differential Harnack estimates for positive solutions to the PME (i.e., (1.9) with $p > 1$). Here $\Delta = \Delta_{g(t)}$ is the Laplace operator with respect to the evolving metric $g(t)$ of the Ricci flow (1.6), and $R = R(x, t)$ is the scalar curvature of $g(t)$. One may check directly that, thanks to the term Ru , any smooth solution u of (1.9) satisfies

$$\frac{d}{dt} \int_M u d\mu = 0,$$

because the volume form evolves under the equation $d(d\mu)/dt = -Rd\mu$. Note that when $p = 1$, (1.9) is simply the conjugate equation (1.8) to the backward heat equation.

When $p > 1$, one usually expresses the PME (1.1) in the following form:

$$\frac{\partial u}{\partial t} = \nabla(u \nabla v) = (p-1)v\Delta u + (p-1)\nabla u \cdot \nabla v, \quad (1.10)$$

where $v = \frac{p}{p-1}u^{p-1}$. It is clear that if u is a positive solution to (1.1), then (1.10) is a parabolic equation, hence the regularity can be studied.

Similarly, with $v = \frac{p}{p-1}u^{p-1}$, (1.9) can be rewritten as

$$\frac{\partial u}{\partial t} = (p-1)v\Delta u + (p-1)\nabla u \cdot \nabla v + Ru, \quad (1.11)$$

which can still be considered formally as the conjugate equation to (1.10) under the Ricci flow if we fix the potential function v . Indeed, one can see that for any smooth functions u and w

$$\begin{aligned} \frac{d}{dt} \int_M u w d\mu &= \int_M (u_t w + u w_t - R u w) d\mu \\ &= \int_M [u_t + (p-1)v\Delta u + (p-1)\nabla u \cdot \nabla v] w d\mu \\ &\quad + \int_M u [w_t - (p-1)v\Delta w - (p-1)\nabla w \cdot \nabla v - R w] d\mu. \end{aligned}$$

Our main result in this paper is the following Aronson-Bénilan and Li-Yau-Hamilton type estimate for smooth positive solutions to the PME (1.9):

Theorem 1.1. *Let $(M^n, g_{ij}(t))$, $t \in [0, T)$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator at each time t . If u is a bounded smooth positive solution to (1.9) with $p > 1$ and $v = \frac{p}{p-1}u^{p-1}$, then we have*

$$\frac{|\nabla v|^2}{v} - 2\frac{v_t}{v} - \frac{R}{v} - \frac{d}{t} \leq 0$$

on $M \times (0, T)$, where $d = \max\{2\alpha, 1\}$ and $\alpha = \frac{n(p-1)}{1+n(p-1)}$.

As immediate consequences of Theorem 1.1, we have the following two versions of Harnack inequalities.

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, if $v_{\min} = \inf_{M \times [0, T]} v > 0$, then for any points $x_1, x_2 \in M$ and $0 < t_1 < t_2 < T$, we have*

$$v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1}\right)^{d/2} \exp\left(\frac{\Gamma}{2v_{\min}}\right), \quad (1.12)$$

where d is the constant in Theorem 1.1, and $\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} (R + \left|\frac{d\gamma}{d\tau}\right|_{g_{ij}(\tau)}^2) d\tau$ with the infimum taking over all smooth curves $\gamma(\tau)$ in M , $t_1 \leq \tau \leq t_2$, so that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

Corollary 1.3. *Under the same assumptions as in Theorem 1.1, for any points $x_1, x_2 \in M$ and $0 < t_1 < t_2 < T$, we have*

$$v(x_1, t_1) - v(x_2, t_2) \leq \frac{d}{2} v_{\max} \ln \frac{t_2}{t_1} + \frac{1}{2} R_{\max} (t_2 - t_1) + \frac{1}{2} \frac{d_{t_1}^2(x_1, x_2)}{t_2 - t_1} \quad (1.13)$$

where d is the constant in Theorem 1.1, $v_{\max} = \sup_{M \times [0, T]} v$, $R_{\max} = \sup_{M \times [0, T]} R$, and $d_{t_1}(x_1, x_2)$ is the distance between x_1, x_2 at time t_1 with respect to the metric $g_{ij}(t_1)$.

More generally, for PME (1.9) with $p > 1$, we actually have

Theorem 1.4. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. If u is a bounded smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1} u^{p-1}$ and any $b \in [1, \infty)$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b-1) \frac{R}{v} - \frac{d}{t} \leq C_0 |b-2| R_{\max}$$

on $M \times (0, T]$, where $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $d = \max\{b\alpha, \frac{b}{2}\}$, $R_{\max} = \sup_{M \times [0, T]} R$, and

$$C_0 = \begin{cases} \frac{2\alpha}{n} + \sqrt{\frac{b\alpha(p-1)}{2}}, & \text{if } b \geq 2 \\ \sqrt{\frac{b\alpha(p-1)(n-1)}{2n}}, & \text{if } 1 \leq b \leq 2. \end{cases}$$

Remark 1.5. *We point out that one can apply similar techniques used in the proof of Theorem 1.4 to obtain Aronson-Bénilan and Li-Yau-Hamilton type estimates for the FDE case, except the calculations are somewhat more involved. Moreover, by applying the same techniques, one can extend the Li-Yau-Hamilton estimate (1.7) of X. Cao and R. Hamilton [5] to the complete noncompact setting. These will be treated elsewhere.*

This paper is organized as follows. In Section 2, we introduce the main quantity, which is involved in our differential Harnack estimates for PME, and compute its evolution under the Ricci flow. In Section 3, we prove Theorem 1.4 for positive solutions to the PME coupled with the Ricci flow and derive the two Harnack inequalities stated in Corollary 1.2 and Corollary 1.3, respectively.

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2. THE MAIN QUANTITY FOR PME AND FDE

In this section, we compute the evolution of certain quantity F introduced below which is essential for deriving our Aronson-Bénilan and Li-Yau-Hamilton estimates in the next section. From now on let $g_{ij}(t)$ be a solution to the Ricci flow (1.6) on $M^n \times [0, T)$ such that $(M^n, g_{ij}(t))$ is a complete Riemannian manifold for each $t \in [0, T)$. Consider the equation

$$\frac{\partial u}{\partial t} = \Delta u^p + aRu, \quad (2.1)$$

where a is an arbitrary constant, $p \in (0, 1) \cup (1, \infty)$, and the Laplacian is with respect to the evolving metric $g_{ij}(t)$. Obviously, (2.1) reduces to (1.9) for $a = 1$.

Let $u > 0$ be a smooth positive solution to (2.1), and set $v = \frac{p}{p-1}u^{p-1}$. It is straightforward to check that

$$\nabla v = pu^{p-2}\nabla u = (p-1)\frac{v}{u}\nabla u, \quad (2.2)$$

and

$$\frac{\partial v}{\partial t} = (p-1)v\Delta v + |\nabla v|^2 + a(p-1)Rv. \quad (2.3)$$

Define

$$\begin{aligned} F &= \frac{|\nabla v|^2}{v} - b\frac{v_t}{v} + c\frac{R}{v} \\ &= -b(p-1)\Delta v + (1-b)\frac{|\nabla v|^2}{v} - ab(p-1)R + c\frac{R}{v}, \end{aligned} \quad (2.4)$$

where b and c are arbitrary constants, and $v_t = \frac{\partial v}{\partial t}$.

We also define operator \mathcal{L} by

$$\mathcal{L} = \frac{\partial}{\partial t} - (p-1)v\Delta.$$

Recall that (cf. [13])

$$\mathcal{L}\left(\frac{f}{g}\right) = \frac{1}{g}\mathcal{L}(f) - \frac{f}{g^2}\mathcal{L}(g) + 2(p-1)v\nabla_i\left(\frac{f}{g}\right)\nabla_i(\ln g). \quad (2.5)$$

Our main purpose in this section is to compute the evolution equation $\mathcal{L}(F)$ of F .

Proposition 2.1. *We have*

$$\begin{aligned}
\mathcal{L}(F) = & 2p\nabla_i F \nabla_i v + \frac{1}{v} \left(c \frac{\partial R}{\partial t} - 2c \nabla_i R \nabla_i v + 2(1-b) R_{ij} \nabla_i v \nabla_j v \right) \\
& - (p-1) \left[(ab+c) \frac{\partial R}{\partial t} - 2a \nabla_i R \nabla_i v + 2R_{ij} \nabla_i v \nabla_j v \right] \\
& - 2(p-1) |\nabla^2 v|^2 - 2b(p-1) R_{ij} \nabla_i \nabla_j v + 2c(p-1) |Rc|^2 \\
& - b(p-1)^2 (\Delta v)^2 + 2(1-b)(p-1) \frac{|\nabla v|^2}{v} \Delta v - ab(p-1)^2 R \Delta v \\
& + a(1-b)(p-1) \frac{|\nabla v|^2}{v} R + (1-b) \frac{|\nabla v|^4}{v^2} + c \frac{|\nabla v|^2}{v^2} R - ac(p-1) \frac{R^2}{v}.
\end{aligned} \tag{2.6}$$

Proof. First of all, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} \Delta v &= 2R_{ij} \nabla_i \nabla_j v + \Delta \frac{\partial v}{\partial t} \\
&= 2R_{ij} \nabla_i \nabla_j v + (p-1)v \Delta^2 v + (p-1)(\Delta v)^2 + 2(p-1) \nabla_i (\Delta v) \nabla_i v \\
&\quad + \Delta |\nabla v|^2 + a(p-1) \Delta(Rv).
\end{aligned}$$

Using the Bochner formula

$$\Delta |\nabla v|^2 = 2 \nabla_i (\Delta v) \nabla_i v + 2 |\nabla^2 v|^2 + 2 R_{ij} \nabla_i v \nabla_j v,$$

one obtains that

$$\begin{aligned}
\mathcal{L}(\Delta v) &= 2p \nabla_i (\Delta v) \nabla_i v + 2R_{ij} \nabla_i \nabla_j v + (p-1)(\Delta v)^2 + 2 |\nabla^2 v|^2 \\
&\quad + 2R_{ij} \nabla_i v \nabla_j v + a(p-1)v \Delta R + 2a(p-1) \nabla_i R \nabla_i v + a(p-1) R \Delta v.
\end{aligned} \tag{2.7}$$

On the other hand, since

$$\begin{aligned}
\mathcal{L}(|\nabla v|^2) &= 2R_{ij} \nabla_i v \nabla_j v + 2 \nabla_i \frac{\partial v}{\partial t} \nabla_i v - (p-1)v \Delta |\nabla v|^2 \\
&= 2R_{ij} \nabla_i v \nabla_j v + 2(p-1) |\nabla v|^2 \Delta v + 2 \nabla_i |\nabla v|^2 \nabla_i v + 2a(p-1)v \nabla_i R \nabla_i v \\
&\quad + 2a(p-1) R |\nabla v|^2 - 2(p-1)v |\nabla^2 v|^2 - 2(p-1)v R_{ij} \nabla_i v \nabla_j v,
\end{aligned}$$

it follows from (2.5) that

$$\begin{aligned}
\mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) &= 2p \nabla_i \frac{|\nabla v|^2}{v} \nabla_i v + \frac{2}{v} R_{ij} \nabla_i v \nabla_j v + 2(p-1) \frac{|\nabla v|^2}{v} \Delta v + \frac{|\nabla v|^4}{v^2} \\
&\quad + 2a(p-1) \nabla_i R \nabla_i v + a(p-1) \frac{|\nabla v|^2}{v} R - 2(p-1) |\nabla^2 v|^2 \\
&\quad - 2(p-1) R_{ij} \nabla_i v \nabla_j v.
\end{aligned} \tag{2.8}$$

Finally, from (2.3) and the evolution equation

$$\frac{\partial R}{\partial t} = \Delta R + 2 |Rc|^2 \tag{2.9}$$

of the scalar curvature R under the Ricci flow, we have

$$\begin{aligned} \mathcal{L}\left(\frac{R}{v}\right) &= 2p\nabla_i \frac{R}{v} \nabla_i v + \frac{|\nabla v|^2}{v^2} R - \frac{2}{v} \nabla_i R \nabla_i v + \frac{1}{v} \frac{\partial R}{\partial t} - a(p-1) \frac{R^2}{v} \\ &\quad - (p-1) \frac{\partial R}{\partial t} + 2(p-1)|Rc|^2. \end{aligned} \quad (2.10)$$

Now, evolution equation (2.6) can be obtained directly from (2.7), (2.8), (2.9), and (2.10). \square

Next, let us take $a = 1$ so that equation (2.1) reduces to our equation (1.9) and further set $c = 1 - b$. Then, we have

$$F = \frac{|\nabla v|^2}{v} - b \frac{v_t}{v} + (1-b) \frac{R}{v} \quad (2.11)$$

$$= y - bz, \quad (2.12)$$

where

$$y = \frac{|\nabla v|^2}{v} + \frac{R}{v} \quad \text{and} \quad z = \frac{v_t}{v} + \frac{R}{v}.$$

Note that in the special case $b = 1$ (hence $c = 0$), we have

$$y - z = -(p-1)\Delta v - (p-1)R$$

in view of (2.3).

For $a = 1$ and $c = 1 - b$ as discussed above, Proposition 2.1 can be further simplified as follows:

Proposition 2.2. *Suppose u is a smooth positive solution to (1.9) and $v = \frac{p}{p-1}u^{p-1}$. Then, for any constant b , the quantity*

$$F = \frac{|\nabla v|^2}{v} - b \frac{v_t}{v} + (1-b) \frac{R}{v}$$

satisfies the equation

$$\begin{aligned} \mathcal{L}(F) &= 2p\nabla_i F \nabla_i v - \left[\frac{b-1}{v} + p-1\right] \left(\frac{\partial R}{\partial t} - 2\nabla_i R \nabla_i v + 2R_{ij} \nabla_i v \nabla_j v\right) \\ &\quad - 2(p-1)|\nabla^2 v + \frac{b}{2}Rc|^2 + \frac{(b-2)^2}{2}(p-1)|Rc|^2 - \frac{1}{b}F^2 \\ &\quad - \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v}\right]F - \frac{b-1}{b}y^2 - \frac{(b-1)(b-2)}{b}y \frac{R}{v}. \end{aligned} \quad (2.13)$$

Proof. It suffices to observe that

$$\begin{aligned}
& -b(p-1)^2(\Delta v)^2 + 2(1-b)(p-1)\frac{|\nabla v|^2}{v}\Delta v - ab(p-1)^2R\Delta v \\
& + a(1-b)(p-1)\frac{|\nabla v|^2}{v}R + (1-b)\frac{|\nabla v|^4}{v^2} + c\frac{|\nabla v|^2}{v^2}R - ac(p-1)\frac{R^2}{v} \\
& = (p-1)\Delta v \cdot F + (1-b)(p-1)\frac{|\nabla v|^2}{v}\Delta v - c(p-1)\frac{R}{v}\Delta v + a(1-b)(p-1)\frac{|\nabla v|^2}{v}R \\
& + (1-b)\frac{|\nabla v|^4}{v^2} + c\frac{|\nabla v|^2}{v^2}R - ac(p-1)\frac{R^2}{v} \\
& = (p-1)\Delta v \cdot F - \frac{1-b}{b}\frac{|\nabla v|^2}{v}F - c(p-1)\frac{R}{v}\Delta v + [1-b + \frac{(1-b)^2}{b}]\frac{|\nabla v|^4}{v^2} \\
& + [c + \frac{c(1-b)}{b}]\frac{|\nabla v|^2}{v^2}R - ac(p-1)\frac{R^2}{v} \\
& = [(p-1)\Delta v - \frac{1-b}{b}\frac{|\nabla v|^2}{v}]F - \frac{c}{b}\frac{R}{v}F - 2c(p-1)\frac{R}{v}\Delta v + [c + \frac{2(1-b)c}{b}]\frac{|\nabla v|^2}{v^2}R \\
& - 2ac(p-1)\frac{R^2}{v} + \frac{c^2}{b}\frac{R^2}{v^2} + \frac{1-b}{b}\frac{|\nabla v|^4}{v^2} \\
& = -\frac{1}{b}[F + ab(p-1)R]F + \frac{2c}{b}\frac{R}{v}F - \frac{c^2}{b}\frac{R^2}{v^2} + c\frac{|\nabla v|^2}{v^2}R + \frac{1-b}{b}\frac{|\nabla v|^4}{v^2} \\
& = -\frac{1}{b}F^2 - [a(p-1)R - \frac{2c}{b}\frac{R}{v}]F - \frac{c^2}{b}\frac{R^2}{v^2} + c\frac{|\nabla v|^2}{v^2}R + \frac{1-b}{b}\frac{|\nabla v|^4}{v^2}.
\end{aligned}$$

Now (2.13) follows easily by plugging in $a = 1$ and $c = 1 - b$. \square

To conclude this section, we recall the following important trace Li-Yau-Hamilton estimate of Hamilton [9] for the Ricci flow which will be crucial in the proofs of our Aronson-Bénilan and Li-Yau-Hamilton estimates in Section 3.

Theorem 2.3 (Hamilton [9]). *Let $(M^n, g_{ij}(t))$, $t \in [0, T)$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator, then for any 1-form V_i on M^n , we have*

$$\frac{\partial R}{\partial t} + 2\nabla_i R V_i + 2R_{ij}V_i V_j + \frac{R}{t} \geq 0$$

for $t \in (0, T)$.

3. ARONSON-BÉNILAN AND LI-YAU-HAMILTON ESTIMATES FOR PME

In this section, we prove Aronson-Bénilan and Li-Yau-Hamilton estimates for smooth positive solutions to PME (1.9) ($p > 1$) under the Ricci flow (1.6) on complete manifold M^n with bounded and nonnegative curvature operator so that Theorem 2.3 applies.

Since M^n may be noncompact, in order to apply the maximum principle argument, we first show a local differential Harnack estimate for $v = \frac{p}{p-1}u^{p-1}$ in some parabolic cylinder $\coprod_{t \in [0, T]} B_t(O, R_0) \times \{t\}$ for any fixed point $O \in M^n$ and any positive constant $R_0 > 0$, where $B_t(O, R_0)$ denotes the geodesic ball centered at O with radius R_0 under

the metric $g_{ij}(t)$. Letting the radius $R_0 \rightarrow \infty$ leads to a rough global differential Harnack estimate, which gives certain bound on the quantity that we are considering. This rough global differential Harnack estimate in turn enables us to adopt a method of Hamilton in [9] to derive the desired estimate. It turns out necessary for us to treat the cases of $b > 2$ and $b < 2$ separately.

3.1. Case 1: $b \geq 2$. In this case, we first need the following local differential Harnack estimate.

Proposition 3.1. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. Suppose u is a smooth positive solution to (1.9) with $p > 1$ and $v = \frac{p}{p-1}u^{p-1}$. Then, for any point $O \in M$ and any constants $R_0 > 0$ and $b \in [2, \infty)$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b-1) \frac{R}{v} - \frac{d}{t} \leq b\alpha \left[\frac{C_1 v_{max}}{R_0^2} + C_2 R_{max} \right] + R_{max}(b-2) \sqrt{\frac{b(p-1)\alpha}{2}}$$

on $\coprod_{t \in (0, T]} B_t(O, R_0) \times \{t\}$, where $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $d = \max\{b\alpha, \frac{b}{2}\}$, $C_1 = (8n + 128 + \frac{16nb^2p^2}{b-1})(p-1)$, $C_2 = 16 + \frac{2(b-2)}{bn}$, $v_{max} = \max_{\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}} v$ and $R_{max} = \sup_{M \times [0, T]} R$.

Proof. Choose a smooth cut-off function $\eta(s)$ defined for $s \geq 0$ such that $\eta(s) = 1$ for $0 \leq s \leq \frac{1}{2}$, $\eta(s) = 0$ for $s \geq 1$, $\eta(s) > 0$ for $\frac{1}{2} < s < 1$, $-16\eta^{\frac{1}{2}} \leq \eta' \leq 0$ and $\eta'' \geq -16\eta \geq -16$. For any fixed point $O \in M$ and any positive number $R_0 > 0$, we define

$$\phi(x, t) = \eta \left(\frac{r(x, t)}{2R_0} \right) \quad (3.1)$$

on $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$, where $r(x, t)$ is the distance function from O at time t .

Set $H = t\phi F - d$ for some constant $d \geq b/2$ to be chosen later, $\tilde{y} = t\phi y$, and $\tilde{z} = t\phi z$. Then, by (2.11) we have

$$t\phi F = \tilde{y} - b\tilde{z},$$

and it follows from Proposition 2.2 that

$$\begin{aligned} & t\phi \mathcal{L}(H) \\ &= t\phi^2 F + t^2 \phi F \phi_t - (p-1)t^2 v \phi F \Delta \phi - 2(p-1)t^2 v \phi \nabla_i \phi \nabla_i F + t^2 \phi^2 \mathcal{L}(F) \\ &= \phi(\tilde{y} - b\tilde{z}) + t\phi_t(\tilde{y} - b\tilde{z}) - (p-1)tv \Delta \phi(\tilde{y} - b\tilde{z}) - 2(p-1)t^2 v \phi \nabla_i \phi \nabla_i F \\ &\quad + 2pt^2 \phi^2 \nabla_i F \nabla_i v - t\phi^2 \left[\frac{b-1}{v} + p-1 \right] Q - (d-\phi)(p-1)t\phi R - (b-1) \left[\frac{2d}{b} - \phi \right] t\phi \frac{R}{v} \\ &\quad - 2(p-1)t^2 \phi^2 |\nabla^2 v + \frac{b}{2} Rc|^2 + \frac{(b-2)^2}{2} (p-1)t^2 \phi^2 |Rc|^2 - \frac{1}{b} (\tilde{y} - b\tilde{z})^2 \\ &\quad - \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v} \right] t\phi H - \frac{b-1}{b} \tilde{y}^2 - \frac{(b-1)(b-2)}{b} \cdot t\phi \frac{R}{v} \tilde{y}, \end{aligned}$$

where we denote by

$$Q = t \frac{\partial R}{\partial t} + R - 2t \nabla_i R \nabla_i v + 2t R_{ij} \nabla_i v \nabla_j v. \quad (3.2)$$

From the definition of ϕ , it is easy to see that

$$|\nabla\phi| = \frac{1}{2R_0}|\eta'| |\nabla r(x, t)| \leq \frac{8}{R_0}\phi^{\frac{1}{2}},$$

and

$$\Delta\phi = \frac{\eta''}{4R_0^2} + \eta' \frac{\Delta r}{2R_0} \geq -\frac{4}{R_0^2} - \frac{8(n-1)}{R_0^2} = -\frac{8n}{R_0^2},$$

where we have used the Laplacian Comparison Theorem for Δr (see e.g. [17]) in the above. On the other hand, since along the Ricci flow we have (see e.g. Lemma B.40 in [7] or Lemma 2.3.3 in [4])

$$\frac{\partial r}{\partial t} = -\inf_{\gamma} \int_{\gamma} Rc(\dot{\gamma}(s), \dot{\gamma}(s)) ds \geq -R_{max}r(x, t),$$

with the infimum taking over all the shortest geodesics connecting O and x at time t , it implies that

$$\frac{\partial\phi}{\partial t} = \frac{\eta'}{2R_0} \frac{\partial r}{\partial t} \leq 16R_{max}.$$

If $H \leq 0$ in $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$, then clearly the estimate that we seek is automatically true. Otherwise, since by definition $H \leq 0$ on the parabolic boundary of $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$, we may assume that H achieves its positive maximum on the parabolic cylinder $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$ at some time $t_0 > 0$ and some interior point x_0 . Then, at (x_0, t_0) , one has

$$\tilde{y} - b\tilde{z} = H(x_0, t_0) + d > 0, \quad F\nabla\phi = -\phi\nabla F, \quad \text{and} \quad \mathcal{L}(H)(x_0, t_0) \geq 0.$$

Moreover, since

$$2pt_0^2\phi^2\nabla_i F\nabla_i v = -2pt_0^2\phi F\nabla_i\phi\nabla_i v \leq 2pt_0^2\phi F|\nabla\phi||\nabla v| \leq \frac{16p}{R_0}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y} - b\tilde{z}),$$

by Theorem 2.3, one has

$$\begin{aligned} & 0 \leq t_0\phi\mathcal{L}(H) \\ & \leq (\tilde{y} - b\tilde{z}) + 16t_0R_{max}(\tilde{y} - b\tilde{z}) + (8n + 128)(p-1)\frac{t_0v}{R_0^2}(\tilde{y} - b\tilde{z}) + \frac{16p}{R_0}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y} - b\tilde{z}) \\ & \quad - \frac{2}{b^2n(p-1)}[\tilde{y} - b\tilde{z} + (b-1)\tilde{y} - \frac{b(b-2)}{2}(p-1)t_0\phi R]^2 + \frac{(b-2)^2}{2}(p-1)t_0^2\phi^2R^2 \\ & \quad - \frac{1}{b}(\tilde{y} - b\tilde{z})^2 \end{aligned}$$

$$\begin{aligned}
&\leq (\tilde{y} - b\tilde{z}) + 16t_0R_{max}(\tilde{y} - b\tilde{z}) + (8n + 128)(p - 1)\frac{t_0v}{R_0^2}(\tilde{y} - b\tilde{z}) + \frac{16p}{R_0}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y} - b\tilde{z}) \\
&\quad - \frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2 - \frac{4(b-1)}{b^2n(p-1)}\tilde{y}(\tilde{y} - b\tilde{z}) + \frac{2(b-2)}{bn}t_0\phi R(\tilde{y} - b\tilde{z}) + \frac{(b-2)^2(p-1)}{2}t_0^2\phi^2R^2 \\
&= (\tilde{y} - b\tilde{z}) \left[-\frac{4(b-1)}{b^2n(p-1)}\tilde{y} + \frac{16p}{R_0}\tilde{y}^{\frac{1}{2}}(t_0v_{max})^{\frac{1}{2}} + (8n + 128)(p-1)\frac{t_0v_{max}}{R_0^2} \right] \\
&\quad + [t_0(16 + \frac{2(b-2)}{bn})R_{max} + 1](\tilde{y} - b\tilde{z}) + \frac{(b-2)^2}{2}(p-1)t_0^2R_{max}^2 - \frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2 \\
&\leq (\tilde{y} - b\tilde{z}) \left[\frac{t_0v_{max}}{R_0^2}(8n + 128 + \frac{16nb^2p^2}{b-1})(p-1) + (16 + \frac{2(b-2)}{bn})t_0R_{max} + 1 \right] \\
&\quad + \frac{(b-2)^2}{2}(p-1)t_0^2R_{max}^2 - \frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2.
\end{aligned}$$

Thus, at (x_0, t_0) ,

$$\begin{aligned}
\tilde{y} - b\tilde{z} &\leq b\alpha \left[\frac{t_0v_{max}}{R_0^2}(8n + 128 + \frac{16nb^2p^2}{b-1})(p-1) + (16 + \frac{2(b-2)}{bn})t_0R_{max} + 1 \right] \\
&\quad + t_0R_{max}(b-2)\sqrt{\frac{b(p-1)\alpha}{2}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
H &\leq b\alpha \left[\frac{t_0v_{max}}{R_0^2}(8n + 128 + \frac{16nb^2p^2}{b-1})(p-1) + (16 + \frac{2(b-2)}{bn})t_0R_{max} + 1 \right] \\
&\quad + t_0R_{max}(b-2)\sqrt{\frac{b(p-1)\alpha}{2}} - d.
\end{aligned}$$

Therefore, for $x \in \coprod_{t \in [0, T]} B_t(x_0, R_0) \times \{t\}$, we have

$$\begin{aligned}
tF &\leq H(x_0, t_0) + d \\
&\leq b\alpha \left[\frac{tv_{max}}{R_0^2}(8n + 128 + \frac{16nb^2p^2}{b-1})(p-1) + (16 + \frac{2(b-2)}{bn})tR_{max} + 1 \right] \\
&\quad + tR_{max}(b-2)\sqrt{\frac{b(p-1)\alpha}{2}}.
\end{aligned}$$

This finishes the proof. \square

If u is bounded on $M \times [0, T]$, then letting $R_0 \rightarrow \infty$, we immediately get

Corollary 3.2. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. Suppose u is a bounded smooth positive solution to (1.9) with $p > 1$. Then, for $v = \frac{p}{p-1}u^{p-1}$ and $b \in [2, \infty)$, we have*

$$\frac{|\nabla v|^2}{v} - b\frac{v_t}{v} - (b-1)\frac{R}{v} - \frac{d}{t} \leq \left[C_2b\alpha + (b-2)\sqrt{\frac{b(p-1)\alpha}{2}} \right] R_{max}$$

on $M \times (0, T]$, where α , d , and R_{max} are the constants in Proposition 3.1.

To further refine the differential Harnack inequality above, we follow a method of Hamilton which uses the following distance-like function (see [9]).

Lemma 3.3. *Let $g_{ij}(t)$, $t \in [0, T]$, be a complete solution to the Ricci flow on M^n with bounded curvature tensor. Then, there exists a smooth function $f(x)$ on M and a positive constant $C > 0$ such that $f \geq 1$, $f(x) \rightarrow \infty$ as $d_0(x, O) \rightarrow \infty$ (for some fixed point $O \in M$),*

$$|\nabla f|_{g(t)} \leq C, \quad \text{and} \quad |\nabla \nabla f|_{g(t)} \leq C$$

on $M \times [0, T]$.

Theorem 3.4. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. If u is a bounded smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1}u^{p-1}$ and $b \in [2, \infty)$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b-1) \frac{R}{v} - \frac{d}{t} \leq C_0(b-2)R_{max}$$

on $M \times (0, T]$, where $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $d = \max\{b\alpha, \frac{b}{2}\}$, $C_0 = \left[\frac{2\alpha}{n} + \sqrt{\frac{b\alpha(p-1)}{2}} \right]$, and $R_{max} = \sup_{M \times [0, T]} R$.

Proof. Let $H = t(F - K) - d$ for $d = \max\{b\alpha, \frac{b}{2}\}$ and some constant $K > 0$ to be determined. Then, from (2.13), we have

$$\begin{aligned} t\mathcal{L}(H) &= t(F - K) + t^2\mathcal{L}(F) \\ &= H + d + 2pt\nabla_i H \nabla_i v - \left[\frac{b-1}{v} + p-1 \right] tQ + tR \left[\frac{b-1}{v} + p-1 \right] \\ &\quad - 2(p-1)t^2|\nabla^2 v + \frac{b}{2}Rc|^2 + \frac{(b-2)^2}{2}(p-1)t^2|Rc|^2 - \frac{1}{b}(H + tK + d)^2 \\ &\quad - t \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v} \right] (H + tK + d) - \frac{b-1}{b} t^2 y^2 - \frac{(b-1)(b-2)}{b} \cdot \frac{R}{v} t^2 y, \end{aligned}$$

where Q is the trace Li-Yau-Hamilton quantity for the evolving scalar curvature defined in (3.2).

By Theorem 2.3, we have

$$\begin{aligned} t\mathcal{L}(H) &\leq 2pt\nabla H \cdot \nabla v - (d-1)(p-1)tR - (b-1)\left(\frac{2d}{b} - 1\right)\frac{tR}{v} + \frac{(b-2)^2}{2}(p-1)t^2R_{max}^2 \\ &\quad - \frac{2}{b^2n(p-1)} \left[H + tK + d + (b-1)ty - \frac{b(b-2)}{2}(p-1)tR \right]^2 \\ &\quad - \frac{1}{b}(H + tK + d)^2 - t \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v} \right] H - \frac{b-1}{b} t^2 y^2 + H + d \\ &\leq 2pt\nabla H \cdot \nabla v - \frac{1}{b\alpha}(H + tK + d)^2 - \frac{4(b-1)}{b^2n(p-1)} ty(H + tK + d) \\ &\quad + \frac{2(b-2)}{bn} tR(H + tK + d) + \frac{(b-2)^2}{2}(p-1)t^2R_{max}^2 \\ &\quad - t \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v} \right] H - \frac{b-1}{b} t^2 y^2 + H + d. \end{aligned}$$

Set $\tilde{H} = H - \epsilon\psi$, where $\psi = e^{At}f$ for some constant A to be determined, and f is the function in Lemma 3.3. Then,

$$\begin{aligned}
& t\mathcal{L}(\tilde{H}) \\
&= t\mathcal{L}(H) - \epsilon At\psi + (p-1)\epsilon tve^{At}(\Delta f) \\
&\leq 2pt\nabla\tilde{H} \cdot \nabla v + 2pt\epsilon e^{At}\nabla f \cdot \nabla v - \frac{1}{b\alpha}(H + tK + d)^2 - \frac{4(b-1)}{b^2n(p-1)}ty(H + tK + d) \\
&\quad + \frac{2(b-2)}{bn}tR(H + tK + d) + \frac{(b-2)^2}{2}(p-1)t^2R_{max}^2 \\
&\quad - t[(p-1)R + \frac{2(b-1)}{b}\frac{R}{v}]\tilde{H} - \frac{b-1}{b}t^2y^2 + H + d - \epsilon At\psi + (p-1)\epsilon tve^{At}(\Delta f).
\end{aligned}$$

From Corollary 3.2, we know that $\tilde{H} < 0$ at $t = 0$ and also outside a fixed compact subset of M for all $t \in (0, T]$. We claim that $\tilde{H} < 0$ on $M^n \times [0, T]$. If not, then $\tilde{H} = 0$ at some first time $t = t_0 > 0$ and at some point $x_0 \in M$. Then, at (x_0, t_0) , we have

$$H = \epsilon\psi, \quad \nabla\tilde{H} = 0, \quad \text{and} \quad \mathcal{L}(\tilde{H}) \geq 0.$$

Thus,

$$\begin{aligned}
0 &\leq t_0\mathcal{L}(\tilde{H}) \\
&< Ct_0\epsilon\psi|\nabla v| - \frac{1}{b\alpha}(\epsilon\psi)^2 - \left(\frac{2d}{b\alpha} - 1\right)\epsilon\psi - \left(\frac{d^2}{b\alpha} - d\right) - \frac{b-1}{b}\frac{t_0^2|\nabla v|^4}{v^2} - At_0\epsilon\psi \\
&\quad + \left[Cv + \frac{2(b-2)}{bn}R_{max}\right]t_0\epsilon\psi - \frac{1}{b\alpha}t_0^2K^2 + \frac{(b-2)^2}{2}(p-1)t_0^2R_{max}^2 \\
&\quad + \frac{2(b-2)}{bn}t_0^2KR_{max} - \frac{2d}{b\alpha}t_0K + \frac{2(b-2)d}{bn}t_0R_{max}.
\end{aligned}$$

Now if we choose $K > 0$ large enough (e.g., $K = [\frac{2\alpha}{n} + \sqrt{\frac{b\alpha(p-1)}{2}}](b-2)R_{max}$) so that

$$-\frac{1}{b\alpha}K^2 + \frac{(b-2)^2}{2}(p-1)R_{max}^2 + \frac{2(b-2)}{bn}KR_{max} \leq 0$$

and

$$-\frac{2d}{b\alpha}K + \frac{2(b-2)d}{bn}R_{max} \leq 0,$$

then we have

$$\begin{aligned}
0 &\leq t_0\mathcal{L}(\tilde{H}) \\
&\leq -\frac{1}{b\alpha}(\epsilon\psi)^2 - \frac{b-1}{b}\frac{t_0^2|\nabla v|^4}{v^2} + \frac{2\sqrt{b-1}}{b\sqrt{\alpha}}\frac{t_0|\nabla v|^2}{v}\epsilon\psi \\
&\quad + \left[\frac{b}{8\sqrt{b-1}}Cv + \frac{2(b-2)}{bn}R_{max}\right]t_0\epsilon\psi - \epsilon At_0\psi.
\end{aligned}$$

It is easy to see that this would lead to a contradiction by furthermore choosing $A > \frac{b}{8\sqrt{b-1}}Cv_{max} + \frac{2(b-2)}{bn}R_{max}$.

Therefore, $\tilde{H} = H - \epsilon e^{At} f < 0$ on $M \times [0, T]$. Letting $\epsilon \rightarrow 0$, it follows that

$$H \leq 0.$$

□

Now, taking $b = 2$ in Theorem 3.4 we immediately get Theorem 1.1.

3.2. Case 2: $1 \leq b \leq 2$. In this case, since $2 - b \geq 0$, we can actually get a simpler local differential Harnack inequality than the one in Proposition 3.1.

Proposition 3.5. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. If u is a smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1} u^{p-1}$, any point $O \in M$, any constants $R_0 > 0$ and $b \in (1, 2]$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b-1) \frac{R}{v} - \frac{d}{t} \leq b\alpha \left[\frac{C_1 v_{max}}{R_0^2} + 8R_{max} \right] + R_{max}(2-b) \sqrt{\frac{b(p-1)\alpha}{2}}$$

on $\coprod_{t \in (0, T]} B_t(O, R_0) \times \{t\}$, where $d = \max\{b\alpha, 1\}$, $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $C_1 = (8n + 128 + \frac{16nb^2p^2}{b-1})(p-1)$, $v_{max} = \max_{\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}} v$ and $R_{max} = \sup_{M \times [0, T]} R$.

Proof. Let $\phi(x, t)$ be the cut-off function defined in (3.1). Denote by $H = t\phi F - d$ for $d \geq 1$, $\tilde{y} = t\phi y$ and $\tilde{z} = t\phi z$. Then

$$t\phi F = \tilde{y} - b\tilde{z},$$

and Proposition 2.2 implies that

$$\begin{aligned} & t\phi \mathcal{L}(H) \\ &= t\phi \mathcal{L}(t\phi F) \\ &= \phi(\tilde{y} - b\tilde{z}) + t\phi_t(\tilde{y} - b\tilde{z}) - (p-1)t^2 v \phi F \Delta \phi - 2(p-1)t^2 v \phi \nabla_i \phi \nabla_i F + 2pt^2 \phi^2 \nabla_i F \nabla_i v \\ & \quad - t\phi^2 \left[\frac{b-1}{v} + p-1 \right] Q + t\phi^2 R \left[\frac{b-1}{v} + p-1 \right] - 2(p-1)t^2 \phi^2 |\nabla^2 v + Rc - \frac{2-b}{2} Rc|^2 \\ & \quad + \frac{(b-2)^2}{2} (p-1)t^2 \phi^2 |Rc|^2 - \frac{1}{b} (\tilde{y} - b\tilde{z})^2 - \left[(p-1)R + \frac{2(b-1)}{b} \frac{R}{v} \right] t\phi(H+d) \\ & \quad - \frac{b-1}{b} \tilde{y}^2 + \frac{(b-1)(2-b)}{b} \cdot \frac{R}{v} t\phi \tilde{y}, \end{aligned}$$

where $Q \geq 0$ is the quantity in (3.2).

Recall that

$$\left| \frac{\partial \phi}{\partial t} \right| \leq 16R_{max}, \quad \Delta \phi \geq -\frac{8n}{R_0^2}, \quad \text{and} \quad |\nabla \phi| \leq \frac{8}{R_0} \phi^{\frac{1}{2}}.$$

If $H \leq 0$ in $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$, then we are done. Otherwise, since $H \leq 0$ on the parabolic boundary of $\coprod_{t \in [0, T]} B_t(O, 2R_0) \times \{t\}$, we may assume that H achieves a positive maximum at time $t_0 > 0$ and some interior point x_0 . Thus, at (x_0, t_0) , we have

$$\tilde{y} - b\tilde{z} = H(x_0, t_0) + d > 0, \quad F \nabla \phi = -\phi \nabla F, \quad \text{and} \quad \mathcal{L}(H)(x_0, t_0) \geq 0.$$

Thus, one has

$$0 \leq t_0 \phi \mathcal{L}(H)$$

$$\begin{aligned}
&\leq (\tilde{y} - b\tilde{z}) + 16t_0 R_{max}(\tilde{y} - b\tilde{z}) + (8n + 128)(p - 1) \frac{t_0 v}{R_0^2}(\tilde{y} - b\tilde{z}) + \frac{16p}{R_0} \tilde{y}^{\frac{1}{2}}(t_0 v)^{\frac{1}{2}}(\tilde{y} - b\tilde{z}) \\
&\quad - \frac{2}{b^2 n(p - 1)}[\tilde{y} - b\tilde{z} + (b - 1)\tilde{y}]^2 + \frac{(b - 2)^2}{2}(p - 1)t_0^2 \phi^2 R^2 - \frac{1}{b}(\tilde{y} - b\tilde{z})^2 \\
&\leq (\tilde{y} - b\tilde{z}) + 16t_0 R_{max}(\tilde{y} - b\tilde{z}) + (8n + 128)(p - 1) \frac{t_0 v}{R_0^2}(\tilde{y} - b\tilde{z}) + \frac{16p}{R_0} \tilde{y}^{\frac{1}{2}}(t_0 v)^{\frac{1}{2}}(\tilde{y} - b\tilde{z}) \\
&\quad - \frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2 - \frac{4(b - 1)}{b^2 n(p - 1)}\tilde{y}(\tilde{y} - b\tilde{z}) + \frac{(b - 2)^2}{2}(p - 1)t_0^2 \phi^2 R_{max}^2 \\
&\leq -\frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2 + (\tilde{y} - b\tilde{z}) \left[-\frac{4(b - 1)}{b^2 n(p - 1)}\tilde{y} + \frac{16p}{R_0} \tilde{y}^{\frac{1}{2}}(t_0 v_{max})^{\frac{1}{2}} + (8n + 128)(p - 1) \frac{t_0 v_{max}}{R_0^2} \right] \\
&\quad + [16t_0 R_{max} + 1](\tilde{y} - b\tilde{z}) + \frac{(b - 2)^2}{2}(p - 1)t_0^2 R_{max}^2 \\
&\leq -\frac{1}{b\alpha}(\tilde{y} - b\tilde{z})^2 + (\tilde{y} - b\tilde{z}) \left[\frac{t_0 v_{max}}{R_0^2}(8n + 128 + \frac{16nb^2 p^2}{b - 1})(p - 1) + 8t_0 R_{max} + 1 \right] \\
&\quad + \frac{(b - 2)^2}{2}(p - 1)t_0^2 R_{max}^2.
\end{aligned}$$

Therefore, at (x_0, t_0) , we have

$$\tilde{y} - b\tilde{z} \leq b\alpha \left[C_1 \frac{t_0 v_{max}}{R_0^2} + 16t_0 R_{max} + 1 \right] + t_0 R_{max}(2 - b) \sqrt{\frac{b(p - 1)\alpha}{2}},$$

where $C_1 = (8n + 128 + \frac{16nb^2 p^2}{b - 1})(p - 1)$. i.e.,

$$H \leq b\alpha \left[C_1 \frac{t_0 v_{max}}{R_0^2} + 16t_0 R_{max} + 1 \right] + t_0 R_{max}(2 - b) \sqrt{\frac{b(p - 1)\alpha}{2}} - d.$$

The Proposition follows immediately since $\phi = 1$ for $x \in \coprod_{t \in [0, T]} B_t(x_0, R_0) \times \{t\}$. \square

For bounded u , letting $R_0 \rightarrow \infty$ in the above Proposition, one gets

Corollary 3.6. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. If u is a bounded smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1}u^{p-1}$ and $b \in (1, 2]$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b - 1) \frac{R}{v} - \frac{d}{t} \leq \left[16b\alpha + (2 - b) \sqrt{\frac{b(p - 1)\alpha}{2}} \right] R_{max}$$

on $M \times (0, T]$, where $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $d = \max\{b\alpha, 1\}$, and $R_{max} = \sup_{M \times [0, T]} R$.

Again, we can refine the above differential Harnack estimate as follows.

Theorem 3.7. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded and nonnegative curvature operator. If u is a bounded smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1}u^{p-1}$ and $b \in (1, 2]$, we have*

$$\frac{|\nabla v|^2}{v} - b \frac{v_t}{v} - (b-1) \frac{R}{v} - \frac{d}{t} \leq C_0(2-b)R_{max}$$

on $M \times (0, T]$, where $\alpha = \frac{bn(p-1)}{2+bn(p-1)}$, $d = \max\{b\alpha, \frac{b}{2}\}$, $C_0 = \sqrt{\frac{b\alpha(p-1)(n-1)}{2n}}$, and $R_{max} = \sup_{M \times [0, T]} R$.

Proof. Let $H = t(F - K) - d$ for $d = \max\{b\alpha, \frac{b}{2}\}$ and some constant $K > 0$ to be determined. From (2.13), we have

$$\begin{aligned} t\mathcal{L}(H) &= t(F - K) + t^2\mathcal{L}(F) \\ &= H + d + 2pt\nabla_i H \nabla_i v - \left[\frac{b-1}{v} + p-1\right]tQ + tR\left[\frac{b-1}{v} + p-1\right] \\ &\quad - 2(p-1)t^2|\nabla^2 v + \frac{b}{2}Rc|^2 + \frac{(b-2)^2}{2}(p-1)t^2|Rc|^2 - \frac{1}{b}(H + tK + d)^2 \\ &\quad - t\left[(p-1)R + \frac{2(b-1)}{b}\frac{R}{v}\right](H + tK + d) - \frac{b-1}{b}t^2y^2 + \frac{(b-1)(2-b)}{b} \cdot \frac{R}{v}t^2y, \end{aligned}$$

where Q is the quantity in (3.2).

Set $\tilde{H} = H - \epsilon\psi$, where $\psi = e^{At}f$ for some constant A to be determined, and f is the function in Lemma 3.3. It then follows from Theorem 2.3 and the choice of d that

$$\begin{aligned} t\mathcal{L}(\tilde{H}) &\leq 2pt\nabla_i \tilde{H} \nabla_i v + 2pt\epsilon e^{At}\nabla_i f \nabla_i v - (d-1)(p-1)tR - \frac{1}{b\alpha}(H + tK + d)^2 \\ &\quad - \frac{4(b-1)}{b^2n(p-1)}ty(H + tK + d) - \frac{2(2-b)}{bn}tR(H + tK + d) \\ &\quad + \frac{(b-2)^2(n-1)}{2n}(p-1)t^2R_{max}^2 - t\left[(p-1)R + \frac{2(b-1)}{b}\frac{R}{v}\right]\tilde{H} - \frac{(b-1)^2}{b\alpha}t^2y^2 \\ &\quad + H + d - \epsilon At\psi + (p-1)\epsilon tve^{At}(\Delta f). \end{aligned}$$

From Corollary 3.6, we know that $\tilde{H} < 0$ at $t = 0$ and outside a fixed compact subset of M for all $t \in (0, T]$. Again we claim $\tilde{H} < 0$ on $M \times [0, T]$. If not, then $\tilde{H} = 0$ at some first time $t_0 > 0$ and some point $x_0 \in M$. Then, at (x_0, t_0) , we have

$$H = \epsilon\psi > 0, \quad \nabla \tilde{H} = 0, \quad \text{and} \quad \mathcal{L}(\tilde{H}) \geq 0.$$

Thus,

$$\begin{aligned}
0 &\leq t_0 \mathcal{L}(\tilde{H}) \\
&\leq 2pt_0 \epsilon e^{At_0} \nabla_i f \nabla_i v - [d + \frac{2(2-b)}{bn(p-1)}d - 1](p-1)t_0 R - \frac{1}{b\alpha}(\epsilon\psi)^2 - \frac{1}{b\alpha}t_0^2 K^2 - \frac{1}{b\alpha}d^2 \\
&\quad - \frac{2d}{b\alpha}\epsilon\psi + \frac{(b-2)^2(n-1)}{2n}(p-1)t_0^2 R_{max}^2 - \frac{(b-1)^2}{b\alpha}t_0^2 y^2 + \epsilon\psi + d \\
&\quad - \epsilon At_0 \psi + (p-1)\epsilon t_0 v e^{At_0}(\Delta f) \\
&< Ct_0 \epsilon \psi |\nabla v| - [(\frac{2-2b}{bn(p-1)} + \frac{1}{\alpha})d - 1]t_0 R - \frac{1}{b\alpha}(\epsilon\psi)^2 - (\frac{2d}{b\alpha} - 1)\epsilon\psi - (\frac{d^2}{b\alpha} - d) \\
&\quad - \frac{(b-1)^2}{b\alpha} \frac{t_0^2 |\nabla v|^4}{v^2} - \epsilon At_0 \psi + Cvt_0 \epsilon \psi - \frac{1}{b\alpha}t_0^2 K^2 + \frac{(b-2)^2(n-1)}{2n}(p-1)t_0^2 R_{max}^2.
\end{aligned}$$

It is not hard to check that with our choices of b and d ,

$$(\frac{2-2b}{bn(p-1)} + \frac{1}{\alpha})d - 1 \geq 0.$$

By setting $K = \sqrt{\frac{b\alpha(p-1)(n-1)}{2n}}(2-b)R_{max}$, we have

$$\begin{aligned}
0 &\leq t_0 \mathcal{L}(\tilde{H}) \\
&\leq -\frac{1}{b\alpha}(\epsilon\psi)^2 - \frac{(b-1)^2}{b\alpha} \frac{t_0^2 |\nabla v|^4}{v^2} + \frac{2(b-1)}{b\alpha} \frac{t_0 |\nabla v|^2}{v} \epsilon\psi + \frac{b}{b-1} Cvt_0 \epsilon \psi - \epsilon At_0 \psi.
\end{aligned}$$

This would lead to a contradiction if we choose $A > \frac{b}{b-1} C v_{max}$.

Therefore we have shown $\tilde{H} = H - \epsilon e^{At} f < 0$ on $M \times [0, T]$. Letting $\epsilon \rightarrow 0$, we get

$$H \leq 0.$$

□

Now, by combining Theorem 3.4 and Theorem 3.7, we get Theorem 1.4.

Moreover, letting $b \rightarrow 1$ in Theorem 3.7, we have

Theorem 3.8. *Let $(M^n, g_{ij}(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow with bounded nonnegative curvature operator. If u is a bounded smooth positive solution to (1.9) with $p > 1$, then for $v = \frac{p}{p-1} u^{p-1}$, we have*

$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} - \frac{d}{t} \leq C_0 R_{max}$$

on $M \times (0, T]$, where $d = \max\{\alpha, \frac{1}{2}\}$, $\alpha = \frac{n(p-1)}{2+n(p-1)}$, $C_0 = \sqrt{\frac{\alpha(p-1)(n-1)}{2n}}$, and $R_{max} = \sup_{M \times [0, T]} R$.

In case $|\nabla v|$ is also bounded, we can improve the coefficient of $\frac{1}{t}$ in Theorem 3.8.

Theorem 3.9. *Suppose that $(M^n, g_{ij}(t))$, $t \in [0, T]$, is a complete solution to the Ricci flow with bounded nonnegative curvature operator. Let u be a smooth positive solution*

to (1.9) for $p > 1$ and $v = \frac{p}{p-1}u^{p-1}$. Assume that both v and $|\nabla v|$ are bounded on $M \times [0, T]$. Then, we have

$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} - \frac{d}{t} \leq C_0 R_{max} \quad (3.3)$$

for all $t \in (0, T]$, where $d = \frac{n(p-1)}{2+n(p-1)}$, $C_0 = \sqrt{\frac{\alpha(p-1)(n-1)}{2n}}$, and $R_{max} = \sup_{M \times [0, T]} R$.

Proof. Let $H = t(F - K) - d$ for $d = \frac{n(p-1)}{2+n(p-1)}$, and some constant $K > 0$ to be determined. From (2.13), we have

$$\begin{aligned} t\mathcal{L}(H) &= 2pt\nabla_i H \nabla_i v - (p-1)tQ + (p-1)tR - 2(p-1)t^2|\nabla^2 v + \frac{1}{2}Rc|^2 \\ &\quad + \frac{1}{2}(p-1)t^2|Rc|^2 - (H + tK + d)^2 - (p-1)tR(H + tK + d) + H + d, \end{aligned}$$

where Q is the quantity in (3.2).

Theorem 2.3 implies that

$$\begin{aligned} t\mathcal{L}(H) &\leq 2pt\nabla_i H \nabla_i v - (d-1)(p-1)tR - \frac{1}{d}(H + tK + d)^2 - \frac{2}{n}tR(H + tK + d) \\ &\quad - \frac{1}{2n}(p-1)t^2R^2 + \frac{1}{2}(p-1)t^2R^2 - (p-1)tRH + H + d. \end{aligned}$$

Now, let $\tilde{H} = H - \epsilon\psi$, where $\psi = e^{At}f$ for some constant A to be determined, and f is the function in Lemma 3.3. Then

$$\begin{aligned} t\mathcal{L}(\tilde{H}) &\leq 2pt\nabla_i \tilde{H} \nabla_i v + 2pt\epsilon e^{At}\nabla_i f \nabla_i v - (d-1)(p-1)tR - \frac{1}{d}(H + tK + d)^2 \\ &\quad - \frac{2}{n}tR(H + tK + d) + \frac{(n-1)}{2n}(p-1)t^2R_{max}^2 - (p-1)tR\tilde{H} + H + d \\ &\quad - \epsilon At\psi + (p-1)\epsilon tve^{At}(\Delta f). \end{aligned}$$

From Corollary 3.6, we know that $\tilde{H} < 0$ at $t = 0$ and outside a fixed compact subset of M for $t \in (0, T]$. Suppose that $\tilde{H} = 0$ for the first time at $t = t_0 > 0$ and some point $x_0 \in M$. Then, at (x_0, t_0) , we have

$$H = \epsilon\psi > 0, \quad \nabla \tilde{H} = 0, \quad \text{and} \quad \mathcal{L}(\tilde{H}) \geq 0.$$

Thus,

$$\begin{aligned} 0 &\leq t_0\mathcal{L}(\tilde{H}) \\ &\leq 2pt_0\epsilon e^{At_0}\nabla_i f \nabla_i v - \frac{1}{d}(\epsilon\psi)^2 - d - \frac{1}{d}t_0^2K^2 - 2\epsilon\psi + \frac{(n-1)}{2n}(p-1)t_0^2R_{max}^2 \\ &\quad + \epsilon\psi + d - \epsilon At_0\psi + (p-1)\epsilon t_0ve^{At_0}(\Delta f) \\ &< Ct_0\epsilon\psi|\nabla v| - \frac{1}{d}(\epsilon\psi)^2 - \epsilon\psi - \epsilon At_0\psi + Cv t_0\epsilon\psi - \frac{1}{d}t_0^2K^2 + \frac{(n-1)}{2n}(p-1)t_0^2R_{max}^2. \end{aligned}$$

Now if we choose $K = \sqrt{\frac{d(p-1)(n-1)}{2n}}R_{max}$, then we get

$$0 \leq t_0\mathcal{L}(\tilde{H}) \leq t_0\epsilon\psi(C|\nabla v| + Cv - A) < 0,$$

provided we choose furthermore $A > C(v_{max} + |\nabla v|_{max})$. Hence, we get a contradiction. Therefore, $\tilde{H} = H - \epsilon e^{At} f < 0$ on $M \times [0, T]$. Letting $\epsilon \rightarrow 0$, we see that $H \leq 0$. \square

Finally, by Integrating the differential Harnack quantity in Theorems 1.4 along space-time curves, we obtain the following Harnack inequalities for v .

Corollary 3.10. *Under the same assumptions as in Theorem 1.4, if moreover, $v_{\min} = \inf_{M \times [0, T]} v > 0$, then for any points $x_1, x_2 \in M$, $0 < t_1 < t_2 \leq T$ and $b \geq 1$, we have*

$$v(x_1, t_1) \leq v(x_2, t_2) \cdot \left(\frac{t_2}{t_1}\right)^{d/b} \exp\left(\frac{\Gamma}{v_{\min}} + \frac{|b-2|}{b} C_0 R_{\max}(t_2 - t_1)\right), \quad (3.4)$$

where d , C_0 and R_{\max} are the constants in Theorem 1.4, and $\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} \left(\frac{b-1}{b} R + \frac{b}{4} \left|\frac{d\gamma}{d\tau}\right|_{g_{ij}(\tau)}^2\right) d\tau$ with the infimum taking over all smooth curves $\gamma(\tau)$ in M , $\tau \in [t_1, t_2]$, so that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

Proof. For any curve $\gamma(\tau)$, $\tau \in [t_1, t_2]$, from x_1 to x_2 , we have

$$\begin{aligned} \log \frac{v(x_2, t_2)}{v(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{d\tau} \log v(\gamma(\tau), \tau) d\tau \\ &= \int_{t_1}^{t_2} \left(\frac{v_\tau}{v} + \frac{\nabla v}{v} \cdot \frac{d\gamma}{d\tau}\right) d\tau \\ &\geq \int_{t_1}^{t_2} \left(\frac{v_\tau}{v} - \frac{|\nabla v|^2}{bv} - \frac{b}{4v} \left|\frac{d\gamma}{d\tau}\right|_{g_{ij}(\tau)}^2\right) d\tau \end{aligned}$$

Theorem 3.4 implies that

$$\log \frac{v(x_2, t_2)}{v(x_1, t_1)} \geq \int_{t_1}^{t_2} \left(-\frac{(b-1)R}{bv_{\min}} - \frac{d}{b\tau} - \frac{|b-2|}{b} C_0 R_{\max} - \frac{b}{4v_{\min}} \left|\frac{d\gamma}{d\tau}\right|_{g_{ij}(\tau)}^2\right) d\tau,$$

which gives (3.4) after exponentiating the both sides. \square

Corollary 3.11. *Under the same assumptions as in Theorem 1.4, for any points $x_1, x_2 \in M$, $0 < t_1 < t_2 \leq T$ and $b \geq 1$, we have*

$$v(x_2, t_2) - v(x_1, t_1) \geq -\frac{d}{b} v_{\max} \ln \frac{t_2}{t_1} - \left(\frac{b-1}{b} + \frac{|b-2|}{b} C_0 v_{\max}\right) R_{\max}(t_2 - t_1) - \frac{b}{4} \frac{d_{t_1}^2(x_1, x_2)}{t_2 - t_1}, \quad (3.5)$$

where d , C_0 and R_{\max} are the constants in Theorem 1.4, and $v_{\max} = \sup_{M \times [0, T]} v$.

Proof. Let $\gamma(\tau)$, $\tau \in [t_1, t_2]$, be a geodesic from x_1 to x_2 at time t_1 with constant speed $\frac{d_{t_1}(x_1, x_2)}{t_2 - t_1}$. Then,

$$\begin{aligned} v(x_2, t_2) - v(x_1, t_1) &= \int_{t_1}^{t_2} \frac{d}{d\tau} v(\gamma(\tau), \tau) d\tau \\ &= \int_{t_1}^{t_2} (v_\tau + (\nabla v, \gamma_\tau)) d\tau \\ &\geq \int_{t_1}^{t_2} (v_\tau - \frac{1}{b} |\nabla v|^2 - \frac{b}{4} \left| \frac{d\gamma}{d\tau} \right|_{g_{ij}(\tau)}^2) d\tau. \end{aligned}$$

Moreover, since the Ricci curvature is positive, the evolving metric is shrinking as time τ increasing. This fact together with Theorem 3.4 implies that

$$v(x_2, t_2) - v(x_1, t_1) \geq \int_{t_1}^{t_2} \left(-\frac{b-1}{b} R - \frac{d}{b\tau} v - \frac{|b-2|}{b} C_0 R_{\max} v - \frac{b}{4} \left| \frac{d\gamma}{d\tau} \right|_{g_{ij}(t_1)}^2 \right) d\tau.$$

□

Obviously, by taking $b = 2$ in Corollaries 3.10 and 3.11, we have Corollaries 1.2 and 1.3, respectively. Moreover, under the assumptions and notations as in Theorem 3.9, one can show that the point-wise Harnack inequalities of v generated by Theorem 3.9 can take the forms of (3.4) and (3.5) with $b = 1$.

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